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Artificial boundary method for the exterior Stokes flow in three dimensions

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SUMMARY

In this paper, the artificial boundary method is considered for the numerical simulation of the exterior Stokes flow in three dimensions. First, an exact relation between the normal stress and the velocity field is obtained on a spherical artificial boundary. With the relation specified on the artificial boundary, the original problem is reduced to a new one only defined on a finite domain. After that, an variational problem equivalent to the reduced problem is derived. By truncating the series term in the formulation, a sequence of approximate variational problems are obtained, which can then be solved with a suitable finite-element scheme. Finally, a numerical example is presented to show the performance of the method. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: exterior problem; artificial boundary condition; Stokes equations

1. INTRODUCTION

Among all the numerical methods for solving the PDEs defined on some unbounded domain, the artificial boundary method may be the most popular one. The key idea of this method is to make the computational domain finite by introducing some artificial boundary and then specify a boundary condition on the artificial boundary to obtain a reduced problem only defined on the finite domain. Generally, this boundary condition should be chosen carefully, so that the reduced problem is not only well posed, but also highly accurate to the original one.

Many mathematicians have developed this method for different problems with different techniques in the last three decades. For example, Engquist and Majda [1], Bayliss and Turkel [2] considered the first-order hyperbolic equations and some other wave-like equations; Han and Wu [3], Yu [4] designed various types of artificial boundary conditions for the exterior

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Laplace equation; Feng [5], Goldstein [6] obtained the non-reflecting boundary conditions for the reduced wave equation; Halpern and Schatzman [7], Han and Bao [8] discussed the incompressible flow in a channel; Grote and Keller [9], Alpert *et al.* [10] considered the exterior problem of time-dependent hyperbolic equation, and so on. In Reference [11], Guirguis obtained a third-order local artificial boundary condition for the exterior Stokes flow on a spherical artificial boundary. There are, to the authors' knowledge, no other reports on this subject.

The main goal of this paper is to develop the artificial boundary method for the exterior Stokes flow in three dimensions. In Section 2, some results on vectorial spherical harmonics will be given. In Section 3, the derivation of an exact artificial boundary condition will be presented step by step. In Section 4, a mixed variational problem is given for the reduced problem after the boundary condition is specified on the spherical artificial boundary. By truncating the series term in the formulation, a sequence of approximate variational problems with increasing accuracy are obtained. These approximate problems can be solved with some suitable numerical scheme. In Section 5, a numerical example is presented to test the performance of the designed method. A conclusion is drawn in the last section.

2. SOME RESULTS ON VECTORIAL SPHERICAL HARMONICS

Throughout this paper, bold symbols are used to denote some vectorial fields or spaces. For any non-negative integer l, denote Y_l^m as the mth spherical harmonic of order l. Then $\{Y_l^m, l\geqslant 0, -l\leqslant m\leqslant l\}$ constitutes an orthogonal basis of space $L^2(S)$ where S denotes the unit spherical surface (see Reference [12, p. 24]). Let

$$H_l^m = r^l Y_l^m, \quad -l\leqslant m\leqslant l$$

then $\mathscr{H}_l = \{H_l^m,\ -l\leqslant m\leqslant l\}$ constitutes a basis of all l-order homogeneous harmonic polynomials. Set

$$\mathscr{I}_l^m \equiv \nabla H_{l+1}^m, \quad l\geqslant 0,\ -(l+1)\leqslant m\leqslant l+1$$

$$\mathscr{T}_l^m \equiv \nabla H_l^m \times \mathbf{x}, \quad l\geqslant 1,\ -l\leqslant m\leqslant l$$

$$\mathscr{N}_l^m \equiv (2l-1)H_{l-1}^m\mathbf{x} - r^2\nabla H_{l-1}^m, \quad l\geqslant 1,\ -(l-1)\leqslant m\leqslant (l-1)$$

where $\mathbf{x}$ is the location vector. In addition, let $\mathbf{I}_l^m$, $\mathbf{T}_l^m$ and $\mathbf{N}_l^m$ be the traces of these vectorial polynomials on S, i.e.

$$\mathbf{I}_l^m = \frac{\mathscr{I}_l^m}{r^l}, \quad \mathbf{T}_l^m = \frac{\mathscr{T}_l^m}{r^l}, \quad \mathbf{N}_l^m = \frac{\mathscr{N}_l^m}{r^l}$$

These functions are called l-order *vectorial spherical harmonics*.

Lemma 2.1

Let $\mathbf{n} = \mathbf{x}/r$ be the unit vector in the radial direction. The following results hold:

$$\nabla\left(\nabla\cdot\frac{\mathbf{I}_l^m}{r^{l+1}}\right) = \frac{(l+1)(2l+1)}{r^{l+3}}\mathbf{N}_{l+2}^m \tag{1}$$

$$\nabla\left(\nabla\cdot\frac{\mathbf{T}_l^m}{r^{l+1}}\right)=0 \tag{2}$$

$$\nabla\left(\nabla\cdot\frac{\mathbf{N}_l^m}{r^{l+1}}\right)=0 \tag{3}$$

$$\left(\nabla\cdot\frac{\mathbf{I}_l^m}{r^{l+1}}\right)\mathbf{n}=-\frac{1}{r^{l+2}}\left\{\frac{(l+1)(2l+1)}{2l+3}\,\mathbf{I}_l^m+\frac{(l+1)(2l+1)}{2l+3}\,\mathbf{N}_{l+2}^m\right\} \tag{4}$$

$$\left(\nabla\cdot\frac{\mathbf{T}_l^m}{r^{l+1}}\right)\mathbf{n}=0 \tag{5}$$

$$\left(\nabla\cdot\frac{\mathbf{N}_l^m}{r^{l+1}}\right)\mathbf{n}=0 \tag{6}$$

$$\nabla\frac{\mathbf{I}_l^m}{r^{l+1}}\cdot\mathbf{n}=-\frac{1}{r^{l+2}}\left\{\frac{1}{2l+3}\,\mathbf{I}_l^m+\frac{(l+1)(2l+1)}{2l+3}\,\mathbf{N}_{l+2}^m\right\} \tag{7}$$

$$\nabla\frac{\mathbf{T}_l^m}{r^{l+1}}\cdot\mathbf{n}=-\frac{1}{r^{l+2}}\,\mathbf{T}_l^m \tag{8}$$

$$\nabla\frac{\mathbf{N}_l^m}{r^{l+1}}\cdot\mathbf{n}=-\frac{l+1}{r^{l+2}}\,\mathbf{N}_l^m \tag{9}$$

The proof of this lemma is given in the appendix.

Lemma 2.2
The families $(\mathbf{I}_l^m,\mathbf{T}_l^m,\mathbf{N}_l^m)$ for all $l\geqslant 0$ form an orthogonal basis of $\mathbf{L}^2(S)$ and

$$\int_S|\mathbf{I}_l^m|^2\,\mathrm{d}s=(l+1)(2l+3),\quad l\geqslant 0$$

$$\int_S|\mathbf{T}_l^m|^2\,\mathrm{d}s=l(l+1),\quad l\geqslant 1$$

$$\int_S|\mathbf{N}_l^m|^2\,\mathrm{d}s=l(2l-1),\quad l\geqslant 1$$

This result can be found in Reference [12, p. 37]

3. EXACT NON-LOCAL ARTIFICIAL BOUNDARY CONDITION

Consider the viscous incompressible flow generated by a body moving with a constant velocity. If the Reynolds number of the flow is small, the fluid motion in the steady state can be approximately described with the exterior problem of the Stokes flow:

$$-\nu\,\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \mathbf{x} \in \Omega_{\mathrm{e}} \tag{10}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{x} \in \Omega_{\mathrm{e}} \tag{11}$$

$$\mathbf{u} = \mathbf{U}, \quad \mathbf{x} \in \Gamma \tag{12}$$

$$\mathbf{u} \to \mathbf{0}, \quad p \to 0, \quad r = |\mathbf{x}| \to +\infty \tag{13}$$

Here Γ is the surface of the moving body; Ω_{e} is the exterior unbounded domain with boundary Γ; ν is the kinematic viscosity of the fluid; $\mathbf{U}$ is the constant velocity of the moving body; $\mathbf{f}$ is the body force function assumed to have compact support, namely, there is a spherical artificial boundary Γ_R with radius $R > 0$ such that $\Gamma_R \subset \Omega_R$ and $\mathrm{supp}(\mathbf{f}) \subset \Omega_R$ if we set $\Omega_R \equiv \Omega_{\mathrm{e}} \cap \{\mathbf{x} \mid |\mathbf{x}| < R\}$. On the domain $\Omega_R^{\mathrm{e}} \equiv \Omega_{\mathrm{e}} \backslash \bar{\Omega}_R$ outside of Γ_R, the following hold

$$-\nu\Delta\mathbf{u} + \nabla p = 0, \quad \mathbf{x} \in \Omega_R^{\mathrm{e}} \tag{14}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{x} \in \Omega_R^{\mathrm{e}} \tag{15}$$

$$\mathbf{u} \to \mathbf{0}, \quad p \to 0, \quad r = |\mathbf{x}| \to +\infty \tag{16}$$

This problem cannot be solved independently since no suitable boundary condition is specified on Γ_R. If the value of the velocity field on Γ_R is given, the solution of problem (14)–(16) can be obtained analytically.

Following the idea of Kelvin (see Reference [13, p. 351]), $\mathbf{u}$ and p are expressed by the summations

$$\mathbf{u} = \sum_{l=0}^{+\infty} \mathbf{u}_l, \quad p = \sum_{l=0}^{+\infty} p_l$$

For each integer $l \geqslant 0$, $(\mathbf{u}_l, p_l)$ are assumed to satisfy equations (14)–(16); moreover, $\mathbf{u}_l$ has the following expression:

$$\mathbf{u}_l = \mathbf{G}_l + c_l(r^2 - R^2)\nabla(\nabla \cdot \mathbf{G}_l) \tag{17}$$

where $\mathbf{G}_l$ is a vectorial harmonic function such that $r^{2l+1}\mathbf{G}_l$ is an l-order homogeneous vectorial harmonic polynomial; c_l is some constant to be determined.

Since

$$\begin{aligned}\nabla \cdot \mathbf{u}_l &= \nabla \cdot \mathbf{G}_l + c_l\left\{2\mathbf{r} \cdot \nabla(\nabla \cdot \mathbf{G}_l) + (r^2 - R^2)\Delta(\nabla \cdot \mathbf{G}_l)\right\} \\ &= \nabla \cdot \mathbf{G}_l + 2c_l\{-(l+2)\}\nabla \cdot \mathbf{G}_l = \{1 - (2l+4)c_l\}\nabla \cdot \mathbf{G}_l\end{aligned} \tag{18}$$

by the incompressible condition (15), c_l is determined as

$$c_l = \frac{1}{2l+4}$$

In addition,

$$\begin{aligned}
\Delta \mathbf{u}_l &= \Delta \mathbf{G}_l + c_l\{\Delta(r^2-R^2)\nabla(\nabla\cdot\mathbf{G}_l)\\
&\quad +2\nabla(r^2-R^2)\cdot\nabla\nabla(\nabla\cdot\mathbf{G}_l)+(r^2-R^2)\Delta\nabla(\nabla\cdot\mathbf{G}_l)\}\\
&= c_l\{6\nabla(\nabla\cdot\mathbf{G}_l)+4\mathbf{r}\cdot\nabla\nabla(\nabla\cdot\mathbf{G}_l)\}\\
&= c_l\{6\nabla(\nabla\cdot\mathbf{G}_l)-4(l+3)\nabla(\nabla\cdot\mathbf{G}_l)\}\\
&= -(4l+6)c_l\nabla(\nabla\cdot\mathbf{G}_l) = -\frac{2l+3}{l+2}\nabla(\nabla\cdot\mathbf{G}_l)
\end{aligned}\tag{19}$$

Substituting Equation (19) into (14) gives

$$\nabla\left(p_l+\nu\,\frac{2l+3}{l+2}\,\nabla\cdot\mathbf{G}_l\right)=0$$

Thus by Condition (16), p_l is determined as

$$p_l = -\nu\,\frac{2l+3}{l+2}\nabla\cdot\mathbf{G}_l \tag{20}$$

By Lemma 2.2, on the artificial boundary Γ_R, $\mathbf{u}(R,\theta,\varphi)$ can be expanded as

$$\mathbf{u}(R,\theta,\varphi)=\sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1}A_l^m\mathbf{I}_l^m+\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}B_l^m\mathbf{T}_l^m+\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}C_l^m\mathbf{N}_l^m$$

where

$$\begin{aligned}
A_l^m &= \frac{1}{(l+1)(2l+3)}\int_S \mathbf{u}(R,\theta,\phi)\cdot\bar{\mathbf{I}}_l^m\,\mathrm{d}s\\
B_l^m &= \frac{1}{l(l+1)}\int_S \mathbf{u}(R,\theta,\phi)\cdot\bar{\mathbf{T}}_l^m\,\mathrm{d}s\\
C_l^m &= \frac{1}{l(2l-1)}\int_S \mathbf{u}(R,\theta,\phi)\cdot\bar{\mathbf{N}}_l^m\,\mathrm{d}s.
\end{aligned}$$

In the following (A_l^m, B_l^m, C_l^m) are called the *Fourier coefficients* of $\mathbf{u}$ on Γ_R. Let

$$\mathbf{G}_l=\begin{cases}\left(\dfrac{R}{r}\right)^{l+1}\displaystyle\sum_{m=-(l+1)}^{l+1}A_l^m\mathbf{I}_l^m, & l=0\\ \left(\dfrac{R}{r}\right)^{l+1}\displaystyle\sum_{m=-(l+1)}^{l+1}A_l^m\mathbf{I}_l^m+\left(\dfrac{R}{r}\right)^{l+1}\sum_{m=-l}^{l}B_l^m\mathbf{T}_l^m+\left(\dfrac{R}{r}\right)^{l+1}\sum_{m=-(l-1)}^{l-1}C_l^m\mathbf{N}_l^m, & l>0\end{cases}\tag{21}$$

It is straightforward to verify that formulae (17) and (20) give the solution of problem (10)–(13) with the boundary value $\mathbf{u}(R,\theta,\varphi)$.

The main goal of the following is to build up a relation between the normal stress and the velocity field on the artificial boundary Γ_R. The stress tensor σ of an incompressible viscous flow with velocity field $\mathbf{u}$ and pressure field p is defined by

$$\sigma(\mathbf{u},p) = -p\mathbf{I} + 2\nu\varepsilon(\mathbf{u})$$

where $\mathbf{I}$ is the second-order unit tensor and

$$\varepsilon(\mathbf{u}) = \frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}}{2}$$

is the rate-of-distortion tensor. Since on Γ_R

$$\begin{aligned}
\nabla\mathbf{u}_l &= \nabla\mathbf{G}_l + c_l\{\nabla(r^2-R^2)\otimes\nabla(\nabla\cdot\mathbf{G}_l) + (r^2-R^2)\nabla\nabla(\nabla\cdot\mathbf{G}_l)\}\\
&= \nabla\mathbf{G}_l + 2c_l\mathbf{r}\otimes\nabla(\nabla\cdot\mathbf{G}_l)\\
2\varepsilon(\mathbf{u}_l) &= \nabla\mathbf{u}_l + (\nabla\mathbf{u}_l)^{\mathrm{T}} = \nabla\mathbf{G}_l + (\nabla\mathbf{G}_l)^{\mathrm{T}} + 2c_l\mathbf{r}\otimes\nabla(\nabla\cdot\mathbf{G}_l) + 2c_l\nabla(\nabla\cdot\mathbf{G}_l)\otimes\mathbf{r}
\end{aligned}$$

thus for Stokes flow, on Γ_R it holds

$$\begin{aligned}
\mathbf{n}\cdot\sigma(\mathbf{u}_l,p_l) &= -p_l\mathbf{n} + \nu\mathbf{n}\cdot 2\varepsilon(\mathbf{u}_l)\\
&= \nu\frac{2l+3}{l+2}(\nabla\cdot\mathbf{G}_l)\mathbf{n} - \frac{(l+1)\nu}{R}\mathbf{G}_l + \nu\nabla\mathbf{G}_l\cdot\mathbf{n}\\
&\quad + 2c_l\nu R\nabla(\nabla\cdot\mathbf{G}_l) - c_l(2l+4)\nu(\nabla\cdot\mathbf{G}_l)\mathbf{n}\\
&= \nu\left\{\frac{l+1}{l+2}(\nabla\cdot\mathbf{G}_l)\mathbf{n} - \frac{(l+1)}{R}\mathbf{G}_l + \frac{R}{l+2}\nabla(\nabla\cdot\mathbf{G}_l) + \nabla\mathbf{G}_l\cdot\mathbf{n}\right\}
\end{aligned}$$

By Lemma 2.1, a simple computation shows that:

1. if $\mathbf{G}_l = \mathbf{I}_l^m/r^{l+1}$, then on Γ_R

$$\begin{aligned}
\mathbf{n}\cdot\sigma(\mathbf{u}_l,p_l) &= \nu\left\{\frac{l+1}{l+2}\left\{-\frac{(l+1)(2l+1)}{2l+3}\frac{\mathbf{I}_l^m}{R^{l+2}} - \frac{(l+1)(2l+1)}{2l+3}\frac{\mathbf{N}_{l+2}^m}{R^{l+2}}\right\}\right.\\
&\quad -\frac{(l+1)}{R^{l+2}}\mathbf{I}_l^m + \frac{1}{l+2}\frac{(l+1)(2l+1)}{R^{l+2}}\mathbf{N}_{l+2}^m\\
&\quad \left. +\frac{1}{R^{l+2}}\left\{-\frac{1}{2l+3}\mathbf{I}_l^m - \frac{(l+1)(2l+1)}{2l+3}\mathbf{N}_{l+2}^m\right\}\right\}\\
&= -\frac{1}{R^{l+2}}\frac{2l^2+4l+3}{l+2}\nu\mathbf{I}_l^m
\end{aligned}$$

2. if $\mathbf{G}_l = \mathbf{T}_l^m / r^{l+1}$, then on Γ_R

$$\mathbf{n}\cdot\sigma(\mathbf{u}_l, p_l) = -\frac{l+2}{R^{l+2}}\, v\mathbf{T}_l^m$$

3. if $\mathbf{G}_l = \mathbf{N}_l^m / r^{l+1}$, then on Γ_R

$$\mathbf{n}\cdot\sigma(\mathbf{u}_l, p_l) = -\frac{2l+2}{R^{l+2}}\, v\mathbf{N}_l^m$$

Compose all terms for $l \geqslant 0$, the normal stress on Γ_R is given by

$$\begin{aligned}\mathbf{n}\cdot\sigma(\mathbf{u}, p) = -\frac{v}{R}\Bigg\{&\sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1}\frac{2l^2+4l+3}{l+2}A_l^m\mathbf{I}_l^m + \sum_{l=1}^{+\infty}\sum_{m=-l}^{l}(l+2)B_l^m\mathbf{T}_l^m\\ &+\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(2l+2)C_l^m\mathbf{N}_l^m\Bigg\} \equiv \mathscr{K}_\infty(\mathbf{u}, p)\end{aligned} \tag{22}$$

This relation can serve as an exact artificial boundary condition on Γ_R. Specifying relation (22) on Γ_R presents a reduced Stokes problem only defined on the bounded domain Ω_R:

$$-v\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \mathbf{x}\in\Omega_R \tag{23}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \mathbf{x}\in\Omega_R \tag{24}$$

$$\mathbf{u} = \mathbf{U}, \quad \mathbf{x}\in\Gamma \tag{25}$$

$$\mathbf{n}\cdot\sigma(\mathbf{u}, p) = \mathscr{K}_\infty(\mathbf{u}, p), \quad x\in\Gamma_R \tag{26}$$

Obviously, the solution of this problem is a restriction of that of problem (10)–(13) to subdomain Ω_R of Ω_{e}.

4. MIXED VARIATIONAL PROBLEM AND ITS APPROXIMATION

Denote

$$\mathbf{V_0} \equiv \{\mathbf{v}\in\mathbf{H}^1(\Omega_R)\,|\,\mathbf{v} = \mathbf{0}, \text{ on } \Gamma\}$$

$$\mathbf{V_U} \equiv \{\mathbf{v}\in\mathbf{H}^1(\Omega_R)\,|\,\mathbf{v} = \mathbf{U}, \text{ on } \Gamma\}$$

$$Q \equiv L^2(\Omega_R)$$

Then problem (23)–(26) is equivalent to the following mixed variational problem:

Find $\mathbf{u}\times p\in\mathbf{V_U}\times Q$ such that

$$\begin{aligned}\mathscr{A}(\mathbf{u},\mathbf{v}) + \mathscr{B}_\infty(\mathbf{u},\mathbf{v}) + \mathscr{C}(\mathbf{v}, p) &= \mathscr{F}(\mathbf{v}), \quad \forall\mathbf{v}\in\mathbf{V_0}\\ \mathscr{C}(\mathbf{u}, q) &= 0, \quad \forall q\in Q\end{aligned} \tag{27}$$

where

$$\mathscr{A}(\mathbf{u},\mathbf{v}) = 2\nu \int_{\Omega_R} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})\,\mathrm{d}\sigma$$

$$\mathscr{B}_\infty(\mathbf{u},\mathbf{v}) = \nu R \left\{ \sum_{l=0}^{+\infty} \sum_{m=-(l+1)}^{l+1} (l+1)(2l+3)\frac{2l^2+4l+3}{l+2} A_l^m \bar{D}_l^m \right.$$
$$\left. + \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} l(l+1)(l+2) B_l^m \bar{E}_l^m + \sum_{l=1}^{+\infty} \sum_{m=-(l-1)}^{l-1} 2l(l+1)(2l-1) C_l^m \bar{F}_l^m \right\}$$

$$\mathscr{C}(\mathbf{v},q) = -\int_{\Omega_R} q \nabla \cdot \mathbf{v}\,\mathrm{d}\sigma$$

$$\mathscr{F}(\mathbf{v}) = \int_{\Omega_R} \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}\sigma, \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega_R),\ \forall \mathbf{v} \in \mathbf{H}^1(\Omega_R),\ \forall q \in L^2(\Omega_R)$$

In the above formulae (A_l^m, B_l^m, C_l^m) and (D_l^m, E_l^m, F_l^m) are the *Fourier coefficients* of vector $\mathbf{u}$ and $\mathbf{v}$ on the artificial boundary Γ_R.

In the numerical implementation, the series term in the expression of the bilinear form $\mathscr{B}_\infty(\mathbf{u},\mathbf{v})$ should be truncated. Denote the resulting approximate bilinear form as

$$\mathscr{B}_N(\mathbf{u},\mathbf{v}) = \nu R \left\{ \sum_{l=0}^{N} \sum_{m=-(l+1)}^{l+1} (l+1)(2l+3)\,\frac{2l^2+4l+3}{l+2} A_l^m \bar{D}_l^m \right.$$
$$\left. + \sum_{l=1}^{\max(1,N)} \sum_{m=-l}^{l} l(l+1)(l+2) B_l^m \bar{E}_l^m + \sum_{l=1}^{N} \sum_{m=-(l-1)}^{l-1} 2l(l+1)(2l-1) C_l^m \bar{F}_l^m \right\}$$

In addition, suppose that $\mathbf{V}_\mathrm{U}^h$ is some discrete finite-element space of $\mathbf{V}_\mathrm{U}$, $\mathbf{V}_\mathbf{0}^h$ is obtained with the similar discretization method as that of $\mathbf{V}_\mathrm{U}^h$ and Q^h is some finite element space of Q. Replacing $\{\mathbf{V}_\mathbf{0}, \mathbf{V}_\mathrm{U}, Q, \mathscr{B}_\infty(\cdot,\cdot)\}$ in mixed variational problem (27) with $\{\mathbf{V}_\mathbf{0}^h, \mathbf{V}_\mathrm{U}^h, Q^h, \mathscr{B}_N(\cdot,\cdot)\}$ presents the following approximate discrete variational problem.

Find $\mathbf{u}_N^h \times P^h \in \mathbf{V}_\mathrm{U}^h \times Q^h$ such that

$$\begin{aligned} &\mathscr{A}(\mathbf{u}_N^h,\mathbf{v}) + \mathscr{B}_N(\mathbf{u}_N^h,\mathbf{v}) + \mathscr{C}(\mathbf{v},p^h) = \mathscr{F}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_\mathbf{0}^h \\ &\mathscr{C}(\mathbf{u}_N^h,q) = 0, \quad \forall q \in Q^h \end{aligned} \tag{28}$$

The solution of this problem is taken as a numerical approximation of problem (10)–(13).

Remark 4.1

$\mathscr{B}_\infty(\cdot,\cdot)$ and $\mathscr{B}_N(\cdot,\cdot)$ are real bilinear forms, though it seems they were complex ones. Since for any $l \geqslant 0$, it holds

$$A_l^{-m} = (-1)^m \bar{A}_l^m; \quad B_l^{-m} = (-1)^m \bar{B}_l^m; \quad C_l^{-m} = (-1)^m \bar{C}_l^m$$
$$D_l^{-m} = (-1)^m \bar{D}_l^m; \quad E_l^{-m} = (-1)^m \bar{E}_l^m; \quad F_l^{-m} = (-1)^m \bar{F}_l^m$$

thus

$$\sum_{m=-(l+1)}^{l+1} A_l^m \bar{D}_l^m = \sum_{m=0}^{l+1}{}' (A_l^m \bar{D}_l^m + \bar{A}_l^m D_l^m) = 2 \sum_{m=0}^{l+1}{}' \operatorname{Real}(A_l^m \bar{D}_l^m)$$

$$\sum_{m=-l}^{l} B_l^m \bar{E}_l^m = \sum_{m=0}^{l}{}' (B_l^m \bar{E}_l^m + \bar{B}_l^m E_l^m) = 2 \sum_{m=0}^{l}{}' \operatorname{Real}(B_l^m \bar{E}_l^m)$$

$$\sum_{m=-(l-1)}^{l-1} C_l^m \bar{F}_l^m = \sum_{m=0}^{l-1}{}' (C_l^m \bar{F}_l^m + \bar{C}_l^m F_l^m) = 2 \sum_{m=0}^{l-1}{}' \operatorname{Real}(C_l^m \bar{F}_l^m)$$

where $'$ denotes a multiplication of factor $\frac{1}{2}$ when $m=0$.

Remark 4.2
Spaces V^h and Q^h should be consistent, namely, the two spaces should satisfy the uniform inf–sup condition.

5. NUMERICAL EXAMPLE

Consider the exterior Stokes flow generated by a sphere of radius a moving with a constant speed U. Assume the viscous and incompressible flow is steady, and then this problem can be formulated as

$$\begin{aligned}
&-v\Delta \mathbf{u} + \nabla p = \mathbf{0}, \quad \mathbf{x} \in \Omega_e \\
&\nabla \cdot \mathbf{u} = 0, \quad \mathbf{x} \in \Omega_e \\
&\mathbf{u} = \mathbf{U}, \quad \mathbf{x} \in \Gamma \\
&\mathbf{u} \to \mathbf{0}, \quad p \to 0, \quad r = |\mathbf{x}| \to +\infty
\end{aligned}$$

where $\Gamma = \{\mathbf{x} | \ |\mathbf{x}| = a\}$; Ω_e is the unbounded domain with boundary of Γ; v is the kinematic viscosity; $\mathbf{U} = (U, 0, 0)$. Under the Cartesian co-ordinates frame $\mathbf{x} = (x_1, x_2, x_3)$, the solution can be expressed by

$$\begin{aligned}
u_1 &= U\left[\frac{3}{4}\frac{a}{r} + \frac{a^3}{4r^3} + \frac{3(r^2 - a^2)ax_1^2}{4r^5}\right] \\
u_2 &= U\left[\frac{3(r^2 - a^2)ax_1x_2}{4r^5}\right] \\
u_3 &= U\left[\frac{3(r^2 - a^2)ax_1x_3}{4r^5}\right] \\
p &= \frac{3}{2} v \frac{Uax_1}{r^3}
\end{aligned}$$

where $r \equiv |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Table I. Mesh information.

Mesh	Node number	Cell number	Mesh size
A	3811	2379	0.5
B	27 791	18 923	0.25

Table II. Maximal relative error of the velocity field on the mesh grids.

Mesh	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
A	0.0558	0.0207	0.0118	0.0102	0.0102
B	0.0564	0.0207	0.0049	0.0034	0.0023

Table III. Maximal relative error of the pressure field on the mesh grids.

Mesh	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
A	0.3918	0.1725	0.1323	0.1267	0.1264
B	0.3646	0.1212	0.0562	0.0486	0.0487

Table IV. Computational drag coefficients. The theoretical value is $(12,0,0)$.

Mesh	N = 2	N = 3	N = 4	N = 5	N = 6
A	11.685	11.968	11.996	11.998	11.998
	−0.076	−0.028	−0.024	−0.024	−0.024
	−0.076	−0.015	−0.013	−0.013	−0.013
B	11.690	11.968	11.996	11.997	11.997
	−0.058	−0.012	−0.007	−0.007	−0.007
	−0.058	−0.017	−0.013	−0.013	−0.013

Now introduce a spherical artificial boundary Γ_R of radius $R=2a$. Since the goal of this example is to test the performance of the artificial boundary condition, it is better to shift the location of the spherical surface to violate the symmetry in the set-up of this problem. For instance, relocate the centre of the spherical surface to point $[d\ d\ d]$, where d is some real number satisfying $|d|<a$. Then

$$\Gamma_R = \{\mathbf{x}=(x_1,x_2,x_3)\,|\,(x_1-d)^2+(x_2-d)^2+(x_3-d)^2=4a^2\}$$

A $P_2\times P_1$ tetrahedron mixed finite element is adopted. In the computation, let $a=1$, $v=1$, $U=1$ and $d=-0.4$. Table I shows some information on the mesh generation. Tables II and III show the maximal relative errors of the velocity field and the pressure field. The different values of N represent the different approximate variational problems. The larger is N, the more accurate is the approximate variational problem. Since the Reynolds number $Re=2aU/v=2$, the theoretical drag coefficient is $C=(12,0,0)$. Table IV shows the computational drag coefficients.

From Tables II and III, it can be observed that when a finer mesh could not present a much more accurate numerical solution, a higher-accuracy approximate variational problem should be used in the computation. For example, when $N=2$, the errors of the computational velocity and pressure fields are almost the same for the two different meshes. But when $N=6$, even with Mesh A, the coarser one, used in the computation, the error of velocity field decreases to $0.0102/0.0558 \approx 18\%$ and the error of pressure field decreases to $0.1264/0.3918 \approx 32\%$.

On the other side, when a higher-accuracy approximate variational problem could not present a much better answer, a finer mesh should be employed. For example, when Mesh A is used, the error of the velocity field for $N=6$ is the same as that for $N=5$. But when Mesh B is used, the error decreases to $0.0023/0.0034 \approx 68\%$ from $N=5$ to 6. The analogous phenomenon can be observed from the computational results of the pressure field. When Mesh A is used in the computation, the error of the pressure field decreases to $0.1323/0.1725 \approx 77\%$ from $N=3$ to 4; but when Mesh B is used, the error decreases to $0.0562/0.1212 \approx 46\%$. A great improvement occurs.

These observations are quite compatible with the following analysis. Since the error of the numerical solution originates from two sources: one is the approximation of the variational problem, the other is the employment of the finite-element scheme. When one is relatively smaller, the other dominates the error.

Another observation is that the computational drag coefficient does not seem too sensitive to the accuracy of the numerical solution. Even a lower-accuracy problem with a relatively coarse mesh can give a quite satisfactory drag coefficient answer.

6. CONCLUSION

The numerical simulation for the exterior Stokes flow with the artificial boundary method has been considered in this paper. An exact relation with the normal stress and the velocity field involved is obtained on an introduced spherical artificial boundary. Imposing this relation on the artificial boundary leads to a reduced problem defined on a finite computational domain. It can then be solved with some numerical scheme. A simple numerical example is presented and its results shows the effectiveness of this method.

APPENDIX: PROOF OF LEMMA 2.1

Proof

From the definition of $\mathscr{I}_l^m$, $\mathscr{T}_l^m$ and $\mathscr{N}_l^m$, it obviously holds

$$\mathbf{x}\cdot\mathscr{I}_l^m=(l+1)H_{l+1}^m; \quad \mathbf{x}\cdot\mathscr{T}_l^m=0; \quad \mathbf{x}\cdot\mathscr{N}_l^m=lr^2H_{l-1}^m$$

$$\nabla\cdot\mathscr{I}_l^m=0; \quad \nabla\cdot\mathscr{T}_l^m=0; \quad \nabla\cdot\mathscr{N}_l^m=l(2l+1)H_{l-1}^m$$

and

$$H_l^m\mathbf{x}=\frac{r^2}{2l+1}\mathscr{I}_{l-1}^m+\frac{1}{2l+1}\mathscr{N}_{l+1}^m$$

Since

$$\nabla \cdot \frac{\mathbf{I}_l^m}{r^{l+1}} = \nabla \cdot \frac{\mathscr{I}_l^m}{r^{2l+1}} = -(2l+1)\frac{1}{r^{2l+3}}\,\mathbf{x} \cdot \mathscr{I}_l^m + \frac{1}{r^{2l+1}}\,\nabla \cdot \mathscr{I}_l^m = -\frac{(l+1)(2l+1)}{r^{2l+3}}\,H_{l+1}^m$$

$$\nabla \cdot \frac{\mathbf{T}_l^m}{r^{l+1}} = \nabla \cdot \frac{\mathscr{T}_l^m}{r^{2l+1}} = -(2l+1)\frac{1}{r^{2l+3}}\,\mathbf{x} \cdot \mathscr{T}_l^m + \frac{1}{r^{2l+1}}\,\nabla \cdot \mathscr{T}_l^m = 0$$

$$\nabla \cdot \frac{\mathbf{N}_l^m}{r^{l+1}} = \nabla \cdot \frac{\mathscr{N}_l^m}{r^{2l+1}} = -(2l+1)\frac{1}{r^{2l+3}}\,\mathbf{x} \cdot \mathscr{N}_l^m + \frac{1}{r^{2l+1}}\,\nabla \cdot \mathscr{N}_l^m = 0$$

formulae (2)–(6) follow. In addition,

$$\begin{aligned}
\nabla\left(\nabla \cdot \frac{\mathbf{I}_l^m}{r^{l+1}}\right) = \nabla\left(\nabla \cdot \frac{\mathscr{I}_l^m}{r^{2l+1}}\right) &= -(l+1)(2l+1)\left\{-(2l+3)\frac{1}{r^{2l+5}}\,H_{l+1}^m\mathbf{x} + \frac{1}{r^{2l+3}}\,\nabla H_{l+1}^m\right\} \\
&= \frac{(l+1)(2l+1)}{r^{2l+5}}\,\{(2l+3)H_{l+1}^m\mathbf{x} - r^2\nabla H_{l+1}^m\} \\
&= \frac{(l+1)(2l+1)}{r^{2l+5}}\,\mathscr{N}_{l+2}^m = \frac{(l+1)(2l+1)}{r^{l+3}}\,\mathbf{N}_{l+2}^m
\end{aligned}$$

Formula (1) holds. Moreover,

$$\begin{aligned}
\nabla\frac{\mathbf{I}_l^m}{r^{l+1}} \cdot \mathbf{x} &= \nabla\frac{\mathscr{I}_l^m}{r^{2l+1}} \cdot \mathbf{x} = \nabla\left(\frac{\mathbf{x} \cdot \mathscr{I}_l^m}{r^{2l+1}}\right) - \frac{\mathscr{I}_l^m}{r^{2l+1}} \\
&= (l+1)\left\{-(2l+1)\,\frac{1}{r^{2l+3}}\,H_{l+1}^m\mathbf{x} + \frac{1}{r^{2l+1}}\,\mathscr{I}_l^m\right\} - \frac{\mathscr{I}_l^m}{r^{2l+1}} \\
&= (l+1)\left\{\frac{2}{2l+3}\,\frac{1}{r^{2l+1}}\,\mathscr{I}_l^m - \frac{2l+1}{2l+3}\frac{1}{r^{2l+3}}\,\mathscr{N}_{l+2}^m\right\} - \frac{\mathscr{I}_l^m}{r^{2l+1}} \\
&= -\frac{1}{r^{l+1}}\left\{\frac{1}{2l+3}\,\mathbf{I}_l^m + \frac{(l+1)(2l+1)}{2l+3}\mathbf{N}_{l+2}^m\right\} \\
\nabla\frac{\mathbf{T}_l^m}{r^{l+1}} \cdot \mathbf{x} &= \nabla\frac{\mathscr{T}_l^m}{r^{2l+1}} \cdot \mathbf{x} = \nabla\left(\frac{\mathbf{x} \cdot \mathscr{T}_l^m}{r^{2l+1}}\right) - \frac{\mathscr{T}_l^m}{r^{2l+1}} = -\frac{\mathbf{T}_l^m}{r^{l+1}} \\
\nabla\frac{\mathbf{N}_l^m}{r^{l+1}} \cdot \mathbf{x} &= \nabla\frac{\mathscr{N}_l^m}{r^{2l+1}} \cdot \mathbf{x} = \nabla\left(\frac{\mathbf{x} \cdot \mathscr{N}_l^m}{r^{2l+1}}\right) - \frac{\mathscr{N}_l^m}{r^{2l+1}} \\
&= l\left(-(2l-1)\frac{1}{r^{2l+1}}H_{l-1}^m\mathbf{x} + \frac{1}{r^{2l-1}}\nabla H_{l-1}^m\right) - \frac{\mathscr{N}_l^m}{r^{2l+1}} = -\frac{l+1}{r^{l+1}}\mathbf{N}_l^m
\end{aligned}$$

formulae (7)–(9) follow. □

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